Q1

1

The first three terms of a geometric sequence are given by $x + 11$, $5x$, and $3x^2$ respectively, where x is a non-zero real number.

Find the value of the sixth term in the sequence, giving your answer as a fraction.

[5]

$u_n = ar^{n-1}$

$r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$

$\frac{5x}{x+11} = \frac{3x^2}{5x}$

$25x^2 = 3x^2(x+11) = 3x^3 + 33x^2$

$0 = 3x^3 + 8x^2$

$0 = x^2(3x+8)$

$x = \cancel{0}, \ -\frac{8}{3}$ ← x must be a non-zero real number, $\therefore \ x = -\frac{8}{3}$

$a = x + 11 = -\frac{8}{3} + 11 = \frac{25}{3}$

$r = \frac{5x}{x+11} = \frac{5\left(-\frac{8}{3}\right)}{\left(-\frac{8}{3}\right)+11} = -\frac{8}{5}$

$u_6 = \left(\frac{25}{3}\right)\left(-\frac{8}{5}\right)^{6-1} = \boxed{-\frac{32768}{375}}$

Q2

2

The sum of the first four terms in a geometric series is 27.2, and the sum of the first eight terms in the same series is 164.9.

Given that the first term of the series is positive, find the common ratio, r, of the series.

[5]

$S_4 = 27.2 = \frac{a(1-r^4)}{1-r} \qquad S_8 = 164.9 = \frac{a(1-r^8)}{1-r} \qquad S_n = \frac{a(1-r^n)}{1-r}$

$\frac{S_8}{S_4} \quad \frac{164.9}{27.2} = \frac{\frac{a(1-r^8)}{1-r}}{\frac{a(1-r^4)}{1-r}} = \frac{\cancel{a}(1-r^8)}{\cancel{1-r}} \times \frac{\cancel{1-r}}{\cancel{a}(1-r^4)}$

$= \frac{1-r^8}{1-r^4} = \frac{\cancel{(1-r^4)}(1+r^4)}{\cancel{1-r^4}}$

$\frac{97}{16} = 1 + r^4$

$97 = 16 + 16r^4$

$81 = 16r^4$

$r = \sqrt[4]{\frac{81}{16}} = \pm\frac{3}{2}$

$a = \frac{27.2(1-r)}{1-r^4}$

For a to be positive, the numerator must be negative, since the denominator is negative for either value of r.

$1 - r < 0 \qquad r = \boxed{\frac{3}{2}}$

Q3

3

A geometric series has first term a, and its terms are connected by the relationship $u_{n+4} = 9u_n$ for all $n \geq 1$.

Given that all the terms of the series are positive, show that the sum of the first twelve terms of the series may be written in the form

$$S_{12} = ka(\sqrt{n} + 1)$$

where k and n are positive integers and $\sqrt{n}$ is a surd.

[4]

$u_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

$u_{n+4} = 9u_n$

$ar^{n+4-1} = ar^{n+3} = 9ar^{n-1}$

$r^{n+3} = 9r^{n-1}$

$r^{\cancel{n}+3-(\cancel{n}-1)} = 9$

$r^4 = 9$

$r = \sqrt{3}$

$S_{12} = \frac{a(1-\sqrt{3}^{12})}{1-\sqrt{3}} = \frac{-728a}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$

$= \frac{-728a(1+\sqrt{3})}{1-3} = \boxed{364a(1+\sqrt{3})}$

Q4a

4a

The first three terms in a geometric progression are $(2k + 6)$, k, $(k - 4)$, where k is a constant.

(a) Find the possible values of k.

[4]

(b) Given that the sum to infinity of the progression exists, find the sum to infinity of the progression.

[4]

a) $r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$

$r = \frac{k}{2k+6} = \frac{k-4}{k}$

$k^2 = (k-4)(2k+6)$

$k^2 = 2k^2 + 6k - 8k - 24$

$0 = k^2 - 2k - 24$

$(k-6)(k+4)$

$\boxed{k = 6, -4}$

Q4b

4b

The first three terms in a geometric progression are $(2k+6)$, k, $(k-4)$, where k is a constant.

(a) Find the possible values of k.

$k = 6, -4$ $\qquad r = \frac{k}{2k+6}$

[4]

(b) Given that the sum to infinity of the progression exists, find the sum to infinity of the progression.

[4]

b) Since S_∞, $-1 < r < 1$ $\qquad S_\infty = \frac{a}{1-r}$

If $k=-4$, $\quad r = \frac{-4}{2(-4)+6} = 2 \rightarrow$ out of range

$\therefore k \neq -4$

If $k = 6$, $\quad r = \frac{6}{2(6)+6} = \frac{1}{3} \rightarrow$ in range

$\therefore k = 6$

$a = 2(6)+6 = 18$, $\qquad r = \frac{1}{3}$

$S_\infty = \frac{18}{1-\frac{1}{3}} = \boxed{27}$

Q5a

5a

The second and third terms of a geometric progression are $(x-1)$ and (x^2-1), where x is a real number not equal to 1 or -1.

$x \neq 1, -1$

Given that the sum to infinity of the progression exists,

(a) find the range of possible values of x.

[5]

Given that the sum to infinity of the progression is -6,

(b) find the two possible values of x.

[4]

a) $-1 < r < 1$

$r = \frac{u_3}{u_2} = \frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$

$-1 < x+1 < 1$

$-2 < x < 0$

However, since $x \neq -1$,

$\boxed{\text{either } -2 < x < -1, \text{ or } -1 < x < 0}$

Q5b

5b

The second and third terms of a geometric progression are $(x-1)$ and (x^2-1), where x is a real number not equal to 1 or -1.

Given that the sum to infinity of the progression exists,

(a) find the range of possible values of x. [5]

$-2 < x < -1, \quad -1 < x < 0 \qquad (r = x+1)$

Given that the sum to infinity of the progression is -6,

(b) find the two possible values of x. [4]

b) $S_\infty = -6 \qquad S_\infty = \frac{a}{1-r}$

$r = x+1$

$a = \frac{u_2}{r} = \frac{x-1}{x+1}$

$S_\infty = -6 = \frac{x-1}{x+1(\cancel{1}-(x+\cancel{1}))} = \frac{x-1}{-x(x+1)} = \frac{x-1}{-x^2-x}$

$-6(-x^2-x) = x-1$

$6x^2+6x = x-1$

$6x^2+5x+1=0$

$(3x+1)(2x+1)=0$

$\boxed{x = -\frac{1}{3}, \ -\frac{1}{2}}$

These two values are within the range found in (a)!

Q6a

6a

The geometric progression S is defined by $S = u_1+u_2+u_3+...+u_n+...$, where u_n denotes the nth term of the progression. The sum to infinity of the progression exists and is denoted by S_∞. The first term of the progression, u_1, is equal to a, and the common ratio of the progression is r_s.

A different progression $T = u_1^2+u_2^2+u_3^2+...+u_n^2+...$ is formed by squaring all the terms of the progression S above. common ratio r_t

(a) Show that $T = u_1^2+u_2^2+u_3^2+...+u_n^2+...$ is also a geometric progression, and that its sum to infinity also exists. [4]

The sum to infinity of the progression T is denoted by T_∞.

(b) Express the ratio $\frac{T_\infty}{S_\infty}$ in terms of a and r, simplifying your answer as far as possible. [3]

(c) Show that if $T_\infty = S_\infty$, then $u_k^2 = u_{2k-1}+u_{2k}$ for all $k \geq 1$. Comment on what this shows about the relationship between the terms of the two progressions. [6]

In a geometric progression: $a+b+c...$, $b^2 = ac$.

a) In S, $\overset{b^2}{u_{k+1}^2} = \overset{ac}{\boxed{u_k u_{k+2}}}$ (for any $k \geq 1$), because S is geometric.

(u_k, u_{k+1}, u_{k+2} are any three consecutive terms in S)

The corresponding terms in T are u_k^2, u_{k+1}^2, u_{k+2}^2.

Now $(\overset{b}{u_{k+1}^2})^2 = (u_k u_{k+2})^2$ substitute from above

$= \overset{a}{u_k^2} \, \overset{c}{u_{k+2}^2}$

$\boxed{\text{Therefore } T \text{ is geometric.}}$

Since S_∞ exists, $-1 < \frac{u_{k+1}}{u_k} < 1 \qquad \left(\text{where } r_s = \frac{u_{k+1}}{u_k}\right)$

It follows that $-1 < \left(\frac{u_{k+1}}{u_k}\right)^2 < 1$

But $\left(\frac{u_{k+1}}{u_k}\right)^2 = \frac{u_{k+1}^2}{u_k^2}$, so

$-1 < \frac{u_{k+1}^2}{u_k^2} < 1 \qquad \left(\text{where } r_t = \frac{u_{k+1}^2}{u_k^2}\right)$

$\boxed{\text{Therefore } T_\infty \text{ exists.}}$

Q6b

6b

The geometric progression S is defined by $S = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$, where u_n denotes the nth term of the progression. The sum to infinity of the progression exists and is denoted by S_∞. The first term of the progression, u_1, is equal to a, and the common ratio of the progression is r.

A different progression $T = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 + \ldots$ is formed by squaring all the terms of the progression S above.

(a) Show that $T = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 + \ldots$ is also a geometric progression, and that its sum to infinity also exists. [4]

The sum to infinity of the progression T is denoted by T_∞.

(b) Express the ratio $\frac{T_\infty}{S_\infty}$ in terms of a and r, simplifying your answer as far as possible. [3]

(c) Show that if $T_\infty = S_\infty$, then $u_k^2 = u_{2k-1} + u_{2k}$ for all $k \geq 1$. Comment on what this shows about the relationship between the terms of the two progressions. [6]

b) In S: a, r $\qquad S_\infty = \frac{a}{1-r}$

In T: a^2, r^2

$S_\infty = \frac{a}{1-r} \qquad T_\infty = \frac{a^2}{1-r^2}$

$$\frac{T_\infty}{S_\infty} = \frac{\frac{a^2}{1-r^2}}{\frac{a}{1-r}} = \frac{a^{\cancel{2}}}{1-r^2} \times \frac{1-r}{\cancel{a}} = \frac{a(1-r)}{1-r^2}$$

$$= \frac{a\cancel{(1-r)}}{\cancel{(1-r)}(1+r)} = \frac{a}{1+r}$$

$$\boxed{\frac{T_\infty}{S_\infty} = \frac{a}{1+r}}$$

Q6c

6c

The geometric progression S is defined by $S = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$, where u_n denotes the nth term of the progression. The sum to infinity of the progression exists and is denoted by S_∞. The first term of the progression, u_1, is equal to a, and the common ratio of the progression is r.

u_4

A different progression $T = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 + \ldots$ is formed by squaring all the terms of the progression S above.

(a) Show that $T = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 + \ldots$ is also a geometric progression, and that its sum to infinity also exists. [4]

The sum to infinity of the progression T is denoted by T_∞.

(b) Express the ratio $\frac{T_\infty}{S_\infty}$ in terms of a and r, simplifying your answer as far as possible.

$\frac{T_\infty}{S_\infty} = \frac{a}{1+r} = 1$ [3]

(c) Show that if $T_\infty = S_\infty$, then $u_k^2 = u_{2k-1} + u_{2k}$ for all $k \geq 1$. Comment on what this shows about the relationship between the terms of the two progressions. [6]

This shows that $u_1^2 = u_1 + u_2$, $u_2^2 = u_3 + u_4$, etc. The first two terms in S are equal to the first term in T. The next two terms in S, are equal to the next term in T, and so on.

c) $a = 1 + r \qquad u_n = ar^{n-1}$

$u_n = ar^{n-1} = (1+r)r^{n-1}$

LHS

$(u_k)^2 = (1+r)^2 r^{2(k-1)} = (1+r)^2 r^{2k-2}$

RHS

$u_{2k-1} = (1+r)r^{2k-1-1} \qquad u_{2k} = (1+r)r^{2k-1}$

$u_{2k-1} + u_{2k} = (1+r)r^{2k-2} + (1+r)r^{2k-1}$

$= (1+r)r^{2k-2} + (1+r)r^{2k-2}r$

$= (1+r)r^{2k-2}(1+r)$

$= (1+r)^2 r^{2k-2}$

$\boxed{\therefore \text{ RHS} = \text{LHS}}$

Each pair of terms in S is equal to a corresponding single term in T.

Q7

7

The kth term of a geometric progression is given by $u_k = (-2)^{k-1}$.

Calculate the sum of the eleventh through twenty-third terms of the sequence whose kth term is given by $v_k = u_k + \sqrt{13}$, where u_k is defined as above. You should give your answer as an exact value.

[5]

$S_{11 \to 23}$

13 terms

This term alone defines the terms in a geometric series, so we need to find the sum of terms 11 to 23 of that geometric series and add $\sqrt{13}$ to each of those 13 terms.

$a = (-2)^{1-1} = 1$

$r = -2$

$S_n = \dfrac{a(r^n - 1)}{r-1}$

$S_{23} - S_{10} = \dfrac{1((-2)^{23} - 1)}{-2-1} - \dfrac{1((-2)^{10} - 1)}{-2-1}$

$= 2796544$

add 13 lots of $\sqrt{13}$

$\boxed{13\sqrt{13} + 2796544}$